A REMARK ON UTILITY STREAMS

Karl Schlechta *
Laboratoire d'Informatique Fondamentale de Marseille †

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We write down in this very short comment some ideas which occured to the author during an email discussion with Kaushik Basu on the paper [BM05].

The author suggested to consider only finite sequences (which solves the embedding problem, as the resulting set is countable), and to compare sequences of unequal length by repeating them until they have the same length, e.g. a sequence of length 2 will be repeated 3 times, and a sequence of legth 3 2 times, and the results will then be compared.

Note that the author discussed somewhat related problems in Section 2.2.7 of [Sch04]. Considering sums, and not only orders, to evaluate sequences is generally difficult, in the sense that often no finite characterizations are possible - see again [Sch04].

We will write down now a few axioms, which seem reasonable, without discussion.

We have a domain X, and consider finite, non-empty sequences, noted σ etc., with values in X, the set of these sequences will be denoted Σ. X has an order $<$, $\equiv$ will express equivalence wrt. this order, and we put restrictions on a resulting order $\prec$ on Σ, with equivalence $\approx$. $\leq$ and $\preceq$ etc. are defined in the obvious way.

Notation 1.1

For σ and σ' of equal length, we write

$\sigma \leq \sigma'$ iff all $\sigma_i \leq \sigma'_i$,

$\sigma < \sigma'$ iff $\sigma \leq \sigma'$ and for one i $\sigma_i < \sigma'_i$, and finally

$\sigma << \sigma'$ iff all $\sigma_i < \sigma'_i$.

The double use of $\leq$ and $<$ will not pose any problem.

*ks@cmi.univ-mrs.fr, karl.schlechta@web.de, http://www.cmi.univ-mrs.fr/ ∼ ks
†UMR 6166, CNRS and Université de Provence, Address: CMI, 39, rue Joliot-Curie, F-13453 Marseille Cedex 13, France

$\{x\}$ is the sequence of length 1.

Concatenation is noted $\circ$. For singletons, we may use simple juxtapposition.

σ^n is σ repeated n times.

Axiom 1.1

(1) Singletons:

(1.1) $x < x' \rightarrow \{x\} \prec \{x'\}$,

(1.2) $x \equiv x' \rightarrow \{x\} \approx \{x'\}$

(essentially Pareto).

(2) concatenation:

(2.1) $\sigma \circ \sigma \approx \sigma$ (this expresses essentially that the mean value is interesting),

(2.2) $\sigma' \prec \sigma'' \rightarrow \sigma \circ \sigma' \prec \sigma \circ \sigma''$,

(2.3) $\sigma' \approx \sigma'' \rightarrow \sigma \circ \sigma' \approx \sigma \circ \sigma''$.

(3) permutation:

$\sigma \circ \sigma' \approx \sigma' \circ \sigma$

(essentially Anonymity).

Fact 1.1

These axioms allow to deduce:

(4) $\sigma \approx \sigma' \rightarrow \sigma \approx \sigma \circ \sigma'$

(5) $\sigma \prec \sigma' \rightarrow \sigma \prec \sigma \circ \sigma'$

(6) if there are $i, j \leq length(\sigma) = length(\sigma')$, $\sigma_i = \sigma'_j$, $\sigma_j = \sigma'_i$, and $\sigma_k = \sigma'_k$ for all other k, then $\sigma \approx \sigma'$ (real Anonymity)

(7) Weak Pareto:

(7.1) $\sigma \leq \sigma' \rightarrow \sigma \preceq \sigma'$,

(7.2) $\sigma < \sigma' \rightarrow \sigma \prec \sigma'$,

(7.3) $\sigma << \sigma' \rightarrow \sigma \prec \sigma'$.

and

(8) to compare sequences of different lengths, in the following sense: When $\prec$ and $\approx$ are defined between σ's of equal length, and Axioms 1-3 hold, then the relation $\prec$ (and $\approx$) is determined for arbitrary sequences.

Proof:

Elementary.

(4) by (2.1) and (2.3).

(5) by (2.1) and (2.2).

(6) Let e.g. σ be $\sigma_0 \circ a \circ \sigma_1 \circ b \circ \sigma_3$, then $ab \approx ba$ by (3), so $\sigma_0 \circ ab \approx \sigma_0 \circ ba$ by (2.3) and (3), so $b \circ \sigma_0 \circ a \approx a \circ \sigma_0 \circ b$ by (3), so $b \circ \sigma_0 \circ a \circ \sigma_1 \approx a \circ \sigma_0 \circ b \circ \sigma_1$ by (2.3), etc.

(7) This follows from (1) and repeated use of (2.2) and (2.3).

(8) Let $m := length(\sigma)$, $n := length(\sigma')$, then we obtain by using (2.1) once, and (2.3) repeatedly, that $\sigma^n \approx \sigma$ and $\sigma'^m \approx \sigma'$, but σ^n and σ'^m have the same length.

□

References

[BM05] K.Basu, T.Mitra: "Possibility theorems for aggregating infinite utility streams equitably", in J.Roemer, K.Suzumura: "Intergenerational equity and sustainability", Palgrave, forthcoming

[Sch04] K.Schlechta: "Coherent Systems", Elsevier, Amsterdam, 2004